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NON-COOPERATIVE STOCHASTIC DOMINANCE GAMES.(U)
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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: Technical Report No. 27
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): NON-COOPERATIVE STOCHASTIC DOMINANCE GAMES
5. TYPE OF REPORT & PERIOD COVERED: Interim
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): Peter C. Fishburn
8. CONTRACT OR GRANT NUMBER(s): N00014-75-C-0857
9. PERFORMING ORGANIZATION NAME AND ADDRESS: The Pennsylvania State University, University Park, Pennsylvania 16802
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 121407
11. CONTROLLING OFFICE NAME AND ADDRESS (code 613B: SJW): Office of Naval Research, Dept. of the Navy, 800 N. Quincy Street, Arlington, Virginia 22217
12. REPORT DATE: November 1977
13. NUMBER OF PAGES: 18
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report): Distribution of this document is unlimited.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):

18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number):
Non-cooperative games
Stochastic dominance
Utility theory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A non-cooperative stochastic dominance game is a non-cooperative game in which the only knowledge about the players' preferences and risk attitudes is presumed to be their preference orders on the set of n-tuples of pure strategies. Stochastic dominance equilibria are defined in terms of mixed strategies for the players that are efficient in the stochastic dominance sense against the strategies of the other players. It is shown that the set of SD equilibria equals all Nash equilibria that can be obtained from combinations of utility functions that are consistent with the → next pg

20. players' known preference orders. The latter part of the report looks at antagonistic stochastic dominance games in which some combination of consistent utility functions is zero-sum over the n-tuples of pure strategies.

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(14) TR-27

(9) TECHNICAL REPORT NO. 27

(6) NON-COOPERATIVE STOCHASTIC DOMINANCE GAMES.

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(11) Nov 77 (12) 21p.

(15)

This research was sponsored by the Office of Naval Research, ONR Contract Number N00014-75-C-0857, ONR Contract Authority identification number NR047-112/11-24-76.

407 732

1. Introduction

This paper investigates equilibrium theory for non-cooperative games in which the information about the players' preferences is confined to their preference orders on the set of pure strategy combinations. Players' risk attitudes, as represented by von Neumann-Morgenstern utility functions [von Neumann and Morgenstern, 1947; Fishburn, 1976], are not presumed to be known in the analysis. We shall assume a finite number $n \geq 2$ of players, indexed by $i = 1,2,\ldots,n$, and use the following notations and assumptions throughout:

X_i, a nonempty finite set of pure strategies for i;

P_i, the set of all mixed strategies (probability distributions on X_i) for i;

$>_i$, an asymmetric weak order for i on $X = \times_{i=1}^{n} X_i$, so that $>_i$ is asymmetric ($x >_i y \Rightarrow$ not $(y >_i x)$) and negatively transitive ($x >_i y$ and $z \in X \Rightarrow x >_i z$ or $z >_i y$), with $x >_i y$ signifying that i prefers x to y;

U_i, the set of all real valued functions on X that preserve $>_i$, so that $u_i \in U_i$ if and only if, for all $x,y \in X$, $x >_i y$ iff $u_i(x) > u_i(y)$.

In addition, $P = \times_{i=1}^{n} P_i$ is the set of all n-tuples of mixed strategies $p = (p_1,\ldots,p_n)$, and u_i is extended from X to P multilinearly as $u_i(p) = \Sigma_X u_i(x_1,\ldots,x_n)p_1(x_1)\ldots p_n(x_n)$. When $u_i \in U_i$ is individual i's von Neumann-Morgenstern utility function on X, his preferences on P are such that he prefers p to q iff $u_i(p) > u_i(q)$.

The usual non-cooperative theory, as presented by Nash [1951] and Luce and Raiffa [1957], among others, assumes the knowledge of all players' risk attitudes. That is, in addition to $(>_1,\ldots,>_n)$, it assumes that each player's von Neumann-Morgenstern utility function $u_i^* \in U_i$ (unique up to positive affine transformations) is known. Although weakening or deletion of this strong and often unrealistic assumption raises many interesting possibilities for analysis, I shall focus here on the case in which $(>_1,\ldots,>_n)$ represents the state of relevant information on the preferences and risk attitudes of the players. Because absence of knowledge about an individual's risk attitudes in the risky decision context corresponds to analysis by first degree stochastic dominance [Whitmore and Findlay, 1977] from an expected-utility viewpoint, I shall refer to a non-cooperative game based on $(>_1,\ldots,>_n)$ as a stochastic dominance game. The relationship between stochastic dominance and equilibria based on $(>_1,\ldots,>_n)$ will become clear as we proceed.

Our focus on $(>_1,\ldots,>_n)$ or on the set $U_1\times\ldots\times U_n$ of all utility functions consistent with $(>_1,\ldots,>_n)$ can be interpreted in several ways. First, if the only available public information on the players' preferences and risk attitudes consists of $(>_1,\ldots,>_n)$, then it seems reasonable to look at the game from this viewpoint so far as a "public" analysis is concerned. Alternatively, if the purpose of the analysis is to examine the collection of all non-cooperative games whose utility-function n-tuples are consistent with $(>_1,\ldots,>_n)$, then $\times U_i$ provides the relevant information base for utilities. Another way to look at our setting is to suppose that every player knows every other player's preferences on X but is ignorant of their risk attitudes and knows that every other player is ignorant of his risk attitudes. Even if every player knows his own risk attitudes, it can be argued that each should use $\times U_i$ as his primary basis for analysis. For example, when $n = 2$, player 1 may presume that player 2's actual information about utilities is $U_1 \times \{u_2^*\}$, but since 1 does not know which $u_2^* \in U_2$ obtains and since 2 knows that 1 does not know which $u_2^* \in U_2$ obtains, both should proceed as if any $(u_1, u_2) \in U_1 \times U_2$ might obtain. This of course need not preclude either player from looking at the $U_1 \times U_2$-based analysis from the perspective of his own true u_i^*, but in doing this he should bear in mind that the other player does not know his u_i^*.

The paper is organized as follows. The next section presents the basic stochastic dominance relations, defines the set $SD(>_1, \ldots, >_n)$ of SD equilibria for the stochastic dominance game as the set of all $p \in P$ such that p_i is SD-efficient against $p_{(i)} = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$, and then proves that the set of SD equilibria equals the union of all Nash equilibria for the different $(u_1, \ldots, u_n) \in \times U_i$. The section concludes with comments on uniform equilibria, which are Nash equilibria for every possible choice of $(u_1, \ldots, u_n)$.

The latter part of the paper looks at antagonistic stochastic dominance games, which are stochastic dominance games in which some $(u_1, \ldots, u_n) \in \times U_i$ has $\Sigma_i u_i(x) = 0$ for all $x \in X$. Section 3 presents a characterization of antagonism in terms of stochastic dominance relations on the set P^+ of all probability distributions on X. It then identifies some aspects of two-person antagonistic games that are similar to aspects of two-person zero-sum games. The final section presents two examples of two-person antagonistic games that illustrate preceding results and definitions.

2. Stochastic Dominance and Equilibria

This section justifies the phrase "stochastic dominance games" by showing that a straightforward application of stochastic dominance concepts to a game whose analysis is based on $(>_1,\ldots,>_n)$ is tantamount to a Nash-equilibrium analysis of all non-cooperative games whose utility functions are congruent with $(>_1,\ldots,>_n)$. The latter part of the section discusses SD equilibria that are Nash equilibria for all possible risk attitudes of the players.

The basic results for stochastic dominance that we shall use will be developed first. Given probability distributions α and β on a nonempty finite set A, and an asymmetric weak order $>_0$ on A, we say that α <u>stochastically dominates</u> β <u>with respect to</u> $>_0$, and write this as $\alpha\ D(>_0)\ \beta$, if and only if for every $y \in A$ the probability that α yields an $x \in A$ worse than y ($y >_0 x$) is no greater than the probability that β yields an $x \in A$ that is worse than y. Thus

$$\alpha\ D(>_0)\ \beta \ \text{ iff } \sum_{\{x:y>_0 x\}} \alpha(x) \leq \sum_{\{x:y>_0 x\}} \beta(x) \text{ for all } y \in A.$$

Strict stochastic dominance is denoted by $SD(>_0)$, with $\alpha\ SD(>_0)\ \beta$ iff $\alpha\ D(>_0)\ \beta$ and not $[\beta\ D(>_0)\ \alpha]$.

Two main lemmas on stochastic dominance will be needed in this section. The first lemma identifies a primary reason for interest in the stochastic dominance idea in connection with expected utility theory. It has been proved by Lehmann [1955], Quirk and Saposnik [1962], and Fishburn [1964], among others. The second lemma is tantamount to Proposition 2 in Fishburn [1974b]. In the lemmas, α, β, α_k, β_k are probability distributions on finite A, $>_0$ is an asymmetric weak order on A, U_0 = {all real valued functions u_0 on A that have $u_0(x) > u_0(y)$ iff $x >_0 y$, for all $x,y \in A$}, and $u_0(\alpha) = \Sigma u_0(x)\alpha(x)$.

Lemma 1: $\alpha\ D(>_0)\ \beta$ <u>iff</u> $u_0(\alpha) \geq u_0(\beta)$ <u>for all</u> $u_0 \in U_0$; $\alpha\ SD(>_0)\ \beta$ <u>iff</u> $u_0(\alpha) > u_0(\beta)$ <u>for all</u> $u_0 \in U_0$.

Lemma 2: *If* $\Sigma_{k=1}^{m} \lambda_k \alpha_k \; SD(>_0) \; \Sigma_{k=1}^{m} \lambda_k \beta_k$ *for no* $(\lambda_1,\ldots,\lambda_m)$ *for which* $\Sigma\lambda_k = 1$ *and* $\lambda_k \geq 0$ *for* $k = 1,\ldots,m$, *then there is a* $u_0 \in U_0$ *such that* $u_0(\beta_k) \geq u_0(\alpha_k)$ *for* $k = 1,\ldots,m$.

In the non-cooperative game context of the present paper, each $p \in P$ induces a probability distribution on X in the natural way, so for all $p,q \in P$ and for all $i \in \{1,\ldots,n\}$ we define $D(>_i)$ by analogy as

$$p \; D(>_i) \; q \text{ iff } \sum_{\{x:y>_i x\}} p_1(x_1)\ldots p_n(x_n) \leq \sum_{\{x:y>_i x\}} q_1(x_1)\ldots,q_n(x_n)$$

$$\text{for all } y \in X,$$

with $p \; SD(>_i) \; q$ iff $p \; D(>_i) \; q$ and not $[q \; D(>_i) \; p]$. For each $p \in P$ and $i \in \{1,\ldots,n\}$ let $p_{(i)} = (p_1,\ldots,p_{i-1},p_{i+1},\ldots,p_n)$ and write p as $(p_i,p_{(i)})$. With $P_{(i)}$ the set of all such $p_{(i)}$, we shall let $S_i(p_{(i)})$ be the set of mixed strategies for player i that are $SD(>_i)$ efficient against $p_{(i)} \in P_{(i)}$, so that

$$p_i \in S_i(p_{(i)}) \text{ iff } (q_i,p_{(i)}) \; SD(>_i) \; (p_i,p_{(i)}) \text{ for no } q_i \in P_i.$$

It is easily seen that $S_i(p_{(i)})$ must contain a pure strategy in P_i and that the correspondence $S_i\colon P_{(i)} \to P_i$ is not generally continuous.

On the basis of the foregoing we shall say that $p \in P$ is an *SD equilibrium* if and only if each p_i is $SD(>_i)$ efficient against $p_{(i)}$. The set $SD(>_1,\ldots,>_n)$ of all SD equilibria for $(>_1,\ldots,>_n)$ is

$$SD(>_1,\ldots,>_n) = \{p \in P\colon p_i \in S_i(p_{(i)}) \text{ for } i = 1,\ldots,n\}.$$

Juxtaposed against this definition, let $S(u_1,\dots,u_n)$ be the set of <u>Nash equilibria</u> in P for $(u_1,\dots,u_n)$ so that

$$S(u_1,\dots,u_n) = \{p \in P\colon u_i(p_i,p_{(i)}) \geq u_i(q_i,p_{(i)}) \text{ for all } q_i \in P_i \text{ and all } i \in \{1,\dots,n\}\}.$$

We know from Nash [1951] that $S(u_1,\dots,u_n)$ is never empty. We know also from Nash [1951], Gale [1953], and Luce and Raiffa [1957], among others, that if $S(u_1,\dots,u_n)$ contains more than one equilibrium, then the elements in $S(u_1,\dots,u_n)$ might not be interchangeable or equivalent, and in fact it is possible to have $u_i(p) > u_i(q)$ for all i when $p,q \in S(u_1,\dots,u_n)$. The difficulties that these phenomena pose for the analysis of non-cooperative games are discussed at length by Luce and Raiffa [Chapter 5, Section 7.8], and I shall not dwell on them here.

The basic relationship between SD equilibria and Nash equilibria is given by the following theorem. This theorem shows that $SD(>_1,\dots,>_n)$ is never empty, and it provides a way of computing all SD equilibria from the Nash equilibria.

Theorem 1: $SD(>_1,\dots,>_n) = \cup\{S(u_1,\dots,u_n)\colon (u_1,\dots,u_n) \in \bigtimes_{i=1}^{n} U_i\}$.

Proof: Suppose first that $p \notin SD(>_1,\dots,>_n)$, so that $p_i \notin S_i(p_{(i)})$ for some i. Given $p_i \notin S_i(p_{(i)})$, Lemma 1 implies that $u_i(q_i,p_{(i)}) > u_i(p)$ for some $q_i \in P_i$ and for all $u_i \in U_i$, so that $p \notin \cup S(u_1,\dots,u_n)$. Therefore $p \in \cup S(u_1,\dots,u_n) \Rightarrow p \in SD(>_1,\dots,>_n)$. Suppose next that $p \in SD(>_1,\dots,>_n)$, so that $p_i \in S_i(p_{(i)})$ for all i, and consider i = 1. Let $X_1 = \{x_{11},x_{12},\dots,x_{1K}\}$, let $p^k = (x_{1k},p_{(1)})$ and $q^k = p$ for $k = 1,\dots,K$--where $(x_{1k},p_{(1)})$ stands for

$(r_1, p_{(1)})$ with $r_1(x_{1k}) = 1$--and let $\lambda_k = q_1(x_{1k})$ for all k. Since $p_1 \in S_1(p_{(1)})$, there is no $q_1 \in P_1$ for which $(q_1, p_{(1)})$ SD($>_1$) p, which is to say that there is no $(\lambda_1, \ldots, \lambda_K) = (q_1(x_{11}), \ldots, q_1(x_{1K}))$ for which $\Sigma\lambda_k p^k$ SD($>_1$) $\Sigma\lambda_k q^k$. Hence, by Lemma 2, there is a $u_1 \in U_1$ for which $u_1(p) \geq u_1(x_1, p_{(1)})$ for all $x_1 \in X_1$, which by linearity implies that $u_1(p) \geq u_1(q_1, p_{(1)})$ for all $q_1 \in P_1$. A similar application of Lemma 2 to each $i > 1$ shows that there is a $u_i \in U_i$ such that $u_i(p) \geq u_i(q_i, p_{(i)})$ for all $q_i \in P_i$. Thus $p \in SD(>_1, \ldots, >_n) \Rightarrow p \in S(u_1, \ldots, u_n)$ for some $(u_1, \ldots, u_n) \in \times U_i$, and the proof is complete.

To illustrate further the connection between stochastic dominance and Nash equilibria we define $p \in P$ as a <u>uniform equilibrium</u> for $(>_1, \ldots, >_n)$ if and only if $p \in S(u_1, \ldots, u_n)$ for all $(u_1, \ldots, u_n) \in \times U_i$. Uniform equilibria are attractive from the stochastic dominance viewpoint since they are Nash equilibria for all possible risk attitudes of the players that are consistent with $(>_1, \ldots, >_n)$. However, uniform equilibria may not exist for an SD game, and even *if they do* exist then some of them may not be desirable from the players' viewpoints. For example, in Nash's sixth example [1951, p. 292], the two-person game with $X_i = \{x_i, y_i\}$ for $i = 1,2$ with $(x_1, x_2) >_i (x_1, y_2) \sim_i (y_1, x_2) \sim_i (y_1, y_2)$ for $i = 1,2$ [where $\sim_i$ denotes indifference for i] has uniform pure equilibria (x_1, x_2) and (y_1, y_2), the latter of which is obviously inferior.

The definition just given says that p is a uniform equilibrium iff $u_i(p) \geq u_i(x_i, p_{(i)})$ for all $u_i \in U_i$, all $x_i \in X_i$ and all $i \in \{1, \ldots, n\}$. Lemma 1 then *implies that* p is a uniform equilibrium iff p $D(>_i)(x_i, p_{(i)})$ for all x_i and all i. The definition of $D(>_i)$ then says *that* p is a uniform equilibrium iff

$$\sum_{\{z: y >_i z\}} p_1(z_1) \ldots p_n(z_n) \leq \min_{X_i} \sum_{\{z_{(i)}: y >_i (x_i, z_{(i)})\}} p_1(z_1) \ldots p_{i-1}(z_{i-1}) p_{i+1}(z_{i+1}) \ldots p_n(z_n)$$

for all $y \in X$ and all i, where $z_{(i)}$ is z with z_i omitted and $(x_i, z_{(i)}) = (z_1, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n)$. The preceding inequality must in fact be an equality, and by looking only at the x_i for which $p_i(x_i) > 0$, it is easily seen that a necessary but not sufficient condition for p to be a uniform equilibrium is that it satisfy the following system of equalities:

$$\sum_{\{z_{(i)}: y >_i (x_i, z_{(i)})\}} p_1(z_1) \ldots p_{i-1}(z_{i-1}) p_{i+1}(z_{i+1}) \ldots p_n(z_n)$$

$$= \sum_{\{z_{(i)}: y >_i (x_i', z_{(i)})\}} p_1(z_1) \ldots p_{i-1}(z_{i-1}) p_{i+1}(z_{i+1}) \ldots p_n(z_n)$$

for all $y \in X$, all $x_i, x_i' \in X_i$ for which $p_i(x_i) p_i(x_i') > 0$, and all $i \in \{1, \ldots, n\}$.

In the two-person game with $X_i = \{x_i, y_i\}$, $(x_1, x_2) \sim_1 (y_1, y_2) >_1 (x_1, y_2) \sim_1 (y_1, x_2)$ and $(x_1, y_2) \sim_2 (y_1, x_2) >_2 (x_1, x_2) \sim_2 (y_1, y_2)$, there is a unique uniform equilibrium that has $p_i(x_i) = p_i(y_i) = 1/2$ for each i. This equilibrium is the only Nash equilibrium for each $(u_1, u_2) \in U_1 \times U_2$ in what amounts to a zero-sum game.

3. Antagonistic Games

The preceding example illustrates a <u>two-person antagonistic game</u>, which is a two-person game in which the players' preference orders on X are the duals (converses, inverses) of each other so that $x >_1 y$ iff $y >_2 x$, for all $x, y \in X$. Since $>_1$ is the dual of $>_2$ if and only if there are $u_1 \in U_1$ and $u_2 \in U_2$ such that $u_1(x) + u_2(x) = 0$ for all $x \in X$, two-person antagonistic games in the $(>_1, >_2)$ setting correspond in a natural way to two-person zero-sum games in the traditional setting. However, since non-constant-sum (u_1, u_2) pairs will exist for any two-person antagonistic game that has $x >_1 y >_1 z$ for some

$x,y,z \in X$, the analysis of such games from the stochastic dominance viewpoint is more involved than the zero-sum analysis. We return to this shortly.

In general I shall use the zero-sum basis to define antagonism and say that $(>_1,\ldots,>_n)$ is <u>antagonistic</u> if and only if there exists $(u_1,\ldots,u_n) \in \times U_i$ for which $\Sigma_i u_i(x) = 0$ for all $x \in X$. For $n \geq 3$ this definition is <u>not</u> tantamount to the requirement that, for all $x,y \in X$, if $x >_i y$ for some i then $y >_j x$ for some $j \neq i$ although every antagonistic $(>_1,\ldots,>_n)$ must satisfy this requirement. For example, if $n = 3$, if $\{a,b,c,x,y,z\} \subseteq X$, and if the three players have preference orders whose restrictions on $\{a,\ldots,z\}$ by decreasing preference are

1. axbycz
2. bzcxay
3. cyazbx,

then $(>_1, >_2, >_3)$ cannot be antagonistic since it must be true that $\Sigma_i[u_i(a) + u_i(b) + u_i(c)] > \Sigma_i[u_i(x) + u_i(y) + u_i(z)]$, but for all $r,s \in \{a,\ldots,z\}$ with $r \neq s$ there are i and j such that $r >_i s$ and $s >_j r$. If α and β are distributions on X with $\alpha(a) = \alpha(b) = \alpha(c) = 1/3$ and $\beta(x) = \beta(y) = \beta(z) = 1/3$, then α SD$(>_i)$ β for $i = 1,2,3$. On the other hand, if $(>_1,\ldots,>_n)$ is antagonistic, then this type of uniform SD result is impossible. This is stated precisely in the following theorem, which characterizes antagonism in terms of stochastic dominance. In the theorem, P^+ denotes the set of all probability distributions on X, and the D$(>_i)$ and SD$(>_i)$ are to be viewed in terms of the definitions of D$(>_0)$ and SD$(>_0)$ near the beginning of section 2.

In general, P^+ will contain many distributions that do not correspond to any $p \in P$. For example, the distribution α on $X = \{x_1,y_1\}\times\{x_2,y_2\}$ that has $\alpha(x_1,x_2) = \alpha(y_1,y_2) = 1/2$ cannot be realized by any $(p_1,p_2) \in P$.

Theorem 2: $(>_1,\ldots,>_n)$ *is antagonistic if and only if there do not exist* $\alpha,\beta \in P^+$ *such that* $\alpha\ D(>_i)\ \beta$ *for all* i *along with* $\alpha\ SD(>_i)\ \beta$ *for some* i.

The proof of this theorem uses a generalization of Lemma 2 whose proof is similar to the proof of Lemma 5 in Fishburn [1974a] and will not be redrawn here. The new lemma is as follows.

Lemma 3: *Suppose that* U *is a nonempty convex cone of real valued functions on* X, *so that* $au + bv \in U$ *whenever* $u,v \in U$ *and* $a,b > 0$, *and that* $>^+$ *on* P^+ *is defined by* $\alpha >^+ \beta$ *iff* $u(\alpha) > u(\beta)$ *for all* $u \in U$, *where* u *on* P^+ *is the linear extension of* u *on* X. *Then for each* $m \in \{1,2,\ldots\}$ *there exists* $(\lambda_1,\ldots,\lambda_m)$ *with* $\lambda_k \geq 0$ *for all* k *and* $\Sigma\lambda_k = 1$ *such that* $\Sigma_1^m \lambda_k \alpha^k >^+ \Sigma_1^m \lambda_k \beta^k$, *where the* α^k *and* β^k *are in* P^+, *if and only if for every* $u \in U$ *there is a* $k \in \{1,\ldots,m\}$ *such that* $u(\alpha^k) > u(\beta^k)$.

Proof of Theorem 2: Suppose first that $(>_1,\ldots,>_n)$ is antagonistic and $\alpha\ D(>_i)\ \beta$ for all i. Then, by Lemma 1, $u_i(\alpha) \geq u_i(\beta)$ for all $u_i \in U_i$ and all i, so that $\Sigma_i u_i(\alpha) \geq \Sigma_i u_i(\beta)$. By the definition of antagonism let the $u_i^0 \in U_i$ satisfy $\Sigma_i u_i^0(x) = 0$ for all $x \in X$. Then $\Sigma_i u_i^0(\alpha) = 0 = \Sigma_i u_i^0(\beta)$, which along with $u_i^0(\alpha) \geq u_i^0(\beta)$ for all i implies that $u_i^0(\alpha) = u_i^0(\beta)$ for all i. The second part of Lemma 1 then says that not $[\alpha\ SD(>_i)\ \beta]$ for every i, and in fact we get both $\alpha\ D(>_i)\ \beta$ and $\beta\ D(>_i)\ \alpha$ for all i.

For the converse proof let $U = \{\Sigma_i u_i\colon\ u_i \in U_i$ for all $i\}$, which is a nonempty convex cone of real valued functions on X, and let $>^+$ on P^+ be as defined in Lemma 3. Now if there exist $\alpha,\beta \in P^+$ such that $u(\alpha) > u(\beta)$ for all $u \in U$, or $\alpha >^+ \beta$, then $u_i(\alpha) \geq u_i(\beta)$ for all $u_i \in U_i$ and all i--hence $\alpha\ D(>_i)\ \beta$ for all i--and, in addition, $u_i(\alpha) > u_i(\beta)$ for some u_i, which by Lemma 1 says that $\alpha\ SD(>_i)\ \beta$ for this i. Hence suppose that there do not exist $\alpha,\beta \in P^+$ such that $\alpha\ D(>_i)\ \beta$ for all i and $\alpha\ SD(>_i)\ \beta$ for some i. Then, as just proved, there do not exist $\alpha,\beta \in P^+$ such that $\alpha >^+ \beta$. Consequently, with $X = \{x^1,\ldots,x^K\}$ and with $[x^k]$ the distribution in P^+ that assigns probability 1 to x^k, there does not exist a K-by-K ρ matrix with $\rho_{jk} \geq 0$ for all $j,k \in \{1,\ldots,K\}$ and $\Sigma_j\Sigma_k\rho_{jk} = 1$ such that $\Sigma_k\Sigma_j\rho_{jk}[x^k] >^+ \Sigma_j\Sigma_k\rho_{jk}[x^j]$. Therefore, by Lemma 3 with $m = K^2$, there exists a $u \in U$ such that $u([x^k]) \leq u([x^j])$ for all j and k. (In other words, with $x^{jk} = x^k$ and $y^{jk} = x^j$, Lemma 3 implies that $u(x^{jk}) \leq u(y^{jk})$ for all (j,k) pairs, so that $u(x^k) \leq u(x^j)$ for all j and k.) Hence $u(x^k) = u(x^j)$ for all j and k so that u is constant on X. Since $u = \Sigma u_i$ for some $u_i \in U_i$, it follows that $(>_1,\ldots,>_n)$ must be antagonistic. This completes the proof.

Just as traditional zero-sum games have features that are not shared by general games, antagonistic games exhibit aspects that are not shared by general non-cooperative games whose analyses are based on $(>_1,\ldots,>_n)$. A case in point is seen from Theorem 2, which says in part that if $(>_1,\ldots,>_n)$ is antagonistic then it cannot be true for $p,q \in P$ that $p\ D(>_i)\ q$ for all i and $p\ SD(>_i)\ q$ for some i. In particular, if $(>_1,\ldots,>_n)$ is antagonistic and $p,q \in SD(>_1,\ldots,>_n)$ then it must be false that $p\ D(>_i)\ q$ for all i and $p\ SD(>_i)\ q$ for some i. On the other hand it is easy to construct games with non-antagonistic $(>_1,\ldots,>_n)$ that have $p,q \in SD(>_1,\ldots,>_n)$ along with $p\ SD(>_i)\ q$ for all i.

There are also important differences between two-person antagonistic games and n-person antagonistic games for $n \geq 3$ that reflect some of the differences between two-person zero-sum games and n-person zero-sum games for $n \geq 3$. We note first some results for two-person antagonistic games.

Theorem 3: Suppose $n = 2$ and $(>_1, >_2)$ is antagonistic. If $e = (e_1, e_2)$ is a pure-strategies equilibrium in $SD(>_1, >_2)$, and if $p \in SD(>_1, >_2)$, then both $p\ SD(>_i)\ e$ and $e\ SD(>_i)\ p$ must be false for $i = 1$ and for $i = 2$. In addition, all pure-strategies equilibria are interchangeable and equivalent. That is, if $e = (e_1, e_2)$ and $e' = (e_1', e_2')$ are pure-strategies equilibria in $SD(>_1, >_2)$, then so are (e_1, e_2') and (e_1', e_2), and $(e_1, e_2) \sim_i (e_1', e_2')$ for $i = 1,2$, where $x \sim_i y$ iff neither $x >_i y$ nor $y >_i x$.

Proof: Suppose with $(>_1, >_2)$ antagonistic that $e \in X$ and $p \in P$ are such that $p\ SD(>_1)\ e$, and that $e, p \in SD(>_1, >_2)$. (By a slight abuse of notation we consider $e \in X$ to be in P also, where $e \in P$ signifies the $(q_1, q_2) \in P$ that has $q_1(e_1) = q_2(e_2) = 1$.) Then Lemma 1 and $p\ SD(>_1)\ e$ imply that $u_1(p) > u_1(e)$ for all $u_1 \in U_1$. Also, since $e \in SD(>_1, >_2)$ and Theorem 1 imply that $u_1(e) \geq u_1(x_1, e_2)$ for all $x_1 \in X_1$ for some (hence for all) $u_1 \in U_1$, we get $u_1(e) \geq u_1(p_1, e_1)$ for all $u_1 \in U_1$. In addition, since $p \in SD(>_1, >_2)$, there exists $u_2 \in U_2$ such that $u_2(p_1, p_2) \geq u_2(p_1, e_2)$ and hence there is a $u_1 \in U_1$, e.g. $u_1 = -u_2$, such that $u_1(p_1, e_2) \geq u_1(p)$. Altogether then the hypotheses of this paragraph imply that there is a $u_1 \in U_1$ for which $u_1(e) \geq u_1(p_1, e_2) \geq u_1(p) > u_1(e)$, which is absurd. Hence $e, p \in SD(>_1, >_2)$ and $(>_1, >_2)$ antagonistic imply not $[p\ SD(>_1)\ e]$. Under the same hypotheses, similar proofs show that none of $p\ SD(>_2)\ e$, $e\ SD(>_1)\ p$ and $e\ SD(>_2)\ p$ can be true. The proof of the second part of Theorem 3 is similar to the simple footnote proof on page 66 in Luce and Raiffa [1957] and will be omitted.

The two-person results of Theorem 3 do not generalize in any straightforward way to antagonistic games with $n \geq 3$. For example, it is easy to construct such games for $n = 3$ that have exactly two pure-strategies equilibria in $SD(>_1,>_2,>_3)$, say (e_1,e_2,e_3) and (e_1',e_2',e_3'), with $e_i \neq e_i'$ for $i = 1,2,3$ and with $e >_1 e'$, $e >_2 e'$ and $e' >_3 e$. The latter imply of course that $e \; SD(>_i) \; e'$ for $i = 1,2$ and that $e' \; SD(>_3) \; e$.

Hence the difficulties encountered with n-person antagonistic games for $n \geq 3$ are not unlike those faced by n-person non-cooperative zero-sum games for $n \geq 3$. But this still leaves open the possibility that two-person antagonistic games have certain nice features beyond those already noted in Theorem 3. As shown in the next section, this possibility is largely unfulfilled.

4. Two-Person Antagonistic Examples

This section presents two examples for two-person antagonistic games that illustrate the theory developed in preceding sections and show that such games lack certain properties that hold for two-person zero-sum games. The first example is a prototypical 2-by-2 game with no saddle point in pure strategies; the other has a pure-strategies equilibrium. In each case $>_1$ is identified by an order-preserving u_1 function on $X = X_1 \times X_2$ that is not intended to reflect the risk attitudes of the two players. For convenience we shall let $a_{ij} = u_1(x)$ when x is composed of the ith pure strategy for player 1 and the jth pure strategy of player 2. Hence each matrix is an a_{ij} matrix. Similarly, $b_{ij} = u_2(x)$ when x is composed of the ith pure strategy for player 1 and the jth pure strategy of player 2. Finally, we shall let λ_i and μ_j be respectively the probability in a mixed strategy for player 1 for his ith pure strategy and the probability in a mixed strategy for player 2 for his jth pure strategy.

Example 1: To illustrate Theorem 1 we consider the following 2-by-2 a_{ij} matrix:

$$\begin{array}{c|cc} & \mu_1 & 1-\mu_1 \\ \hline \lambda_1 & 2 & 0 \\ 1-\lambda_1 & 1 & 3 \end{array}$$

For convenience we shall denote a mixed strategy for player 1 as λ_1 instead of $(\lambda_1, 1-\lambda_1)$, and similarly for the second player. By the stochastic dominance definitions of section 2, we have $\lambda_1 \in S_1(\mu_1)$ iff (λ_1', μ_1) SD$(>_1)$ (λ_1, μ_1) for no $\lambda_1' \in [0,1]$. Using the definition immediately following Lemma 2, $\lambda_1 \in S_1(\mu_1)$ iff, for each $\lambda_1' \in [0,1]$,

either $(1-\mu_1)(\lambda_1' - \lambda_1) > 0$, [y has $a_{ij} = 1$]

or $(1-\mu_1)(\lambda_1' - \lambda_1) + \mu_1(\lambda_1 - \lambda_1') > 0$, [y has $a_{ij} = 2$]

or $(1-\mu_1)(\lambda_1' - \lambda_1) + \mu_1(\lambda_1 - \lambda_1') + \mu_1(\lambda_1' - \lambda_1) > 0$, [y has $a_{ij} = 3$]

or equality holds in place of > in the preceding three lines.

Inspection of this system then shows that

$$\begin{aligned} S_1(\mu_1) &= \{0\} && \text{if } \mu_1 \le 1/2 \\ S_1(\mu_1) &= [0,1] && \text{if } 1/2 < \mu_1 < 1 \\ S_1(\mu_1) &= \{1\} && \text{if } \mu_1 = 1, \end{aligned}$$

and a similar analysis for the efficient sets for player 2 gives

$$\begin{aligned} S_2(\lambda_1) &= \{1\} && \text{if } \lambda_1 = 0 \\ S_2(\lambda_1) &= [0,1] && \text{if } 0 < \lambda_1 < 1 \\ S_2(\lambda_1) &= \{0\} && \text{if } \lambda_1 = 1. \end{aligned}$$

Therefore $\lambda_1 \in S_1(\mu_1)$ and $\mu_1 \in S_2(\lambda_1)$ iff $0 < \lambda_1 < 1$ and $1/2 < \mu_1 < 1$, so that $SD(>_1,>_2) = (0,1)\times(1/2,1)$. In other words, $p \in SD(>_1,>_2)$ if and only if player 1 uses some probability strictly between 0 and 1 for his first pure strategy and player 2 uses some probability strictly between 1/2 and 1 for his first pure strategy.

Consider next the determination of Nash equilibria for various (u_1,u_2) pairs. For convenience we normalize the u_i functions between 0 and 1 with $a_{12} = b_{22} = 0$, $a_{22} = b_{12} = 1$, $1 > a_{11} > a_{21} > 0$ and $1 > b_{21} > b_{11} > 0$. The expected utilities are $u_1(\lambda_1,\mu_1) = \mu_1\lambda_1 a_{11} + \mu_1(1 - \lambda_1)a_{21} + (1 - \mu_1)(1 - \lambda_1)$ and $u_2(\lambda_1,\mu_1) = \mu_1\lambda_1 b_{11} + \mu_1(1 - \lambda_1)b_{21} + (1 - \mu_1)\lambda_1$. For a particular choice of the a_{ij} and b_{ij}, the pair (λ_1,μ_1) is a Nash equilibrium iff $u_1(\lambda_1,\mu_1) \geq \mu_1(\lambda_1',\mu_1)$ for all $\lambda_1' \in [0,1]$ and $u_2(\lambda_1,\mu_1) \geq u_2(\lambda_1,\mu_1')$ for all $\mu_1' \in [0,1]$. Since it is clear that there are no such equilibria for λ_1 or μ_1 in $\{0,1\}$, assume that $0 < \lambda_1 < 1$ and $0 < \mu_1 < 1$. Reduction of the preceding inequalities then shows that $(\lambda_1,\mu_1) \in US(u_1,u_2)$ if and only if the a_{ij} and b_{ij} can be chosen such that

$$\mu_1 = 1/(1 + a_{11} - a_{21})$$
$$\lambda_1 = b_{21}/(1 + b_{21} - b_{11}),$$

which under the restrictions $1 > a_{11} > a_{21} > 0$ and $1 > b_{21} > b_{11} > 0$ gives $US(u_1,u_2) = (0,1)\times(1/2,1)$. This is of course the same result arrived at for $SD(>_1,>_2)$ in the preceding paragraph, as required by Theorem 1.

One further point for this example is in order, and that is that the set of all zero-sum equilibria in $US(u_1,-u_1)$ is a proper subset of $SD(>_1,>_2)$. In the present case $US(u_1,-u_1)$ is all (λ_1,μ_1) of the form $((1 - a_{21})/(1 + a_{11} - a_{21}),$

$1/(1 + a_{11} - a_{21}))$ with $1 > a_{11} > a_{21} > 0$, which is the union of $(t/(1 + t), 1/(1 + t))\times\{1/(1 + t)\}$ over all t strictly between 0 and 1. Thus there will often be $p \in SD(>_1,>_2)$ that are Nash equilibria for (u_1,u_2) only when (u_1,u_2) is not constant or zero-sum.

Example 2: The a_{ij} matrix for our second two-person antagonistic example is

	μ_1	μ_2	μ_3	μ_4	μ_5
λ_1	6	1	3	3	2
λ_2	1	6	3	3	2
λ_3	3	3	5	0	2
λ_4	3	3	0	5	2
λ_5	4	4	4	4	3

Normalizing again so that max $a_{ij} = 1$, min $a_{ij} = 0$, and similarly for the b_{ij}, we require

$$1 = a_{11} > a_{33} > a_{51} > a_{55} > a_{15} > a_{21} > a_{34} = 0,$$
$$1 = b_{34} > b_{21} > b_{15} > b_{55} > b_{51} > b_{33} > b_{11} = 0.$$

The equilibria in $SD(>_1,>_2)$ include p, q and e where

$$p \text{ has } \lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1/2,$$
$$q \text{ has } \lambda_3 = \lambda_4 = \mu_3 = \mu_4 = 1/2,$$
$$e \text{ has } \lambda_5 = \mu_5 = 1.$$

The pure-strategies equilibrium e is a uniform equilibrium, but p is in $S(u_1,u_2)$

iff $1 + a_{21} \geq 2a_{51}$ and $b_{21} \geq 2b_{15}$, and q is in $S(u_1,u_2)$ iff $a_{33} \geq 2a_{51}$ and $1 + b_{33} \geq 2b_{15}$.

According to Theorem 3, $SD(>_i)$ cannot hold between e and p or between e and q in either direction. However, since $a_{11} > a_{33}$ and $a_{21} > a_{34}$, p $SD(>_1)$ q. Similarly q $SD(>_2)$ p. Hence even when a two-person antagonistic game has a saddle point, stochastic dominance relationships can hold between SD equilibria of the game.

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